Class 10: Best-fitting ellipsoid

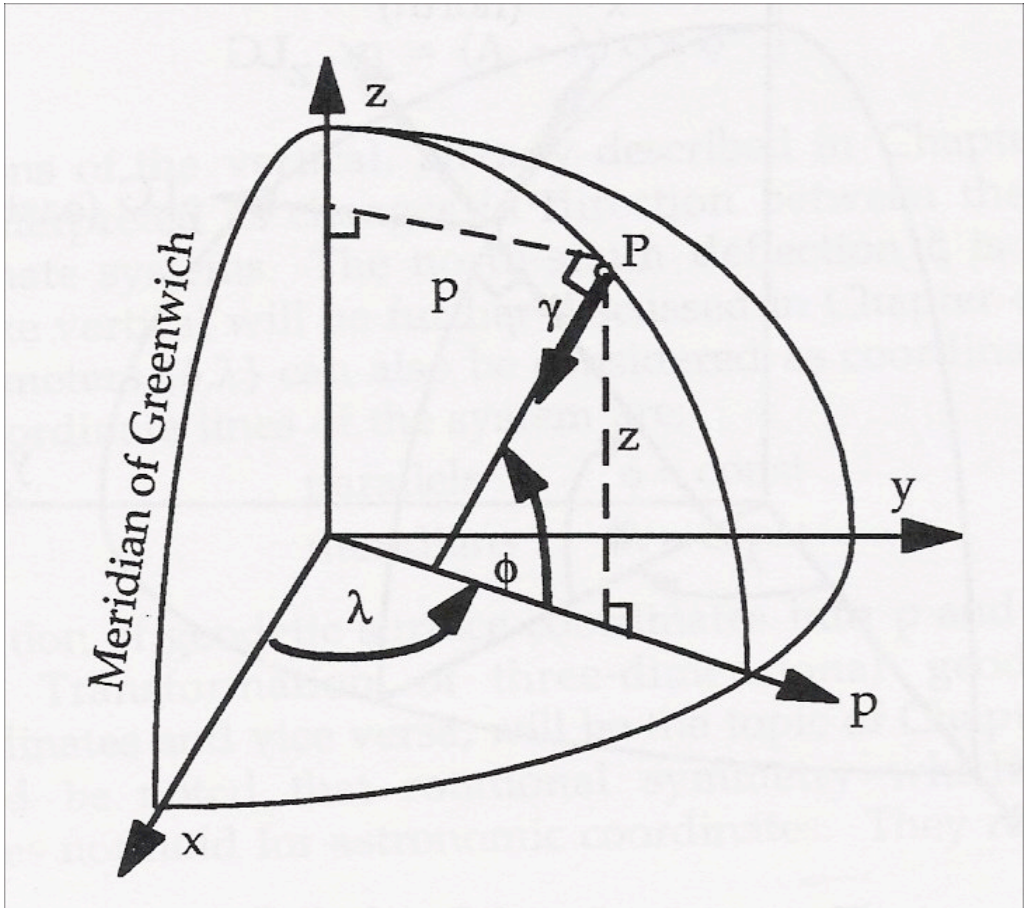


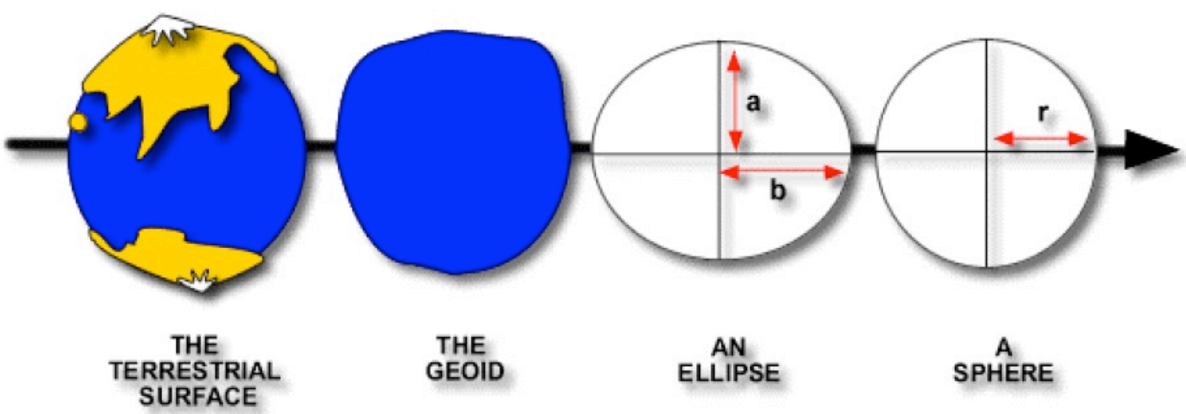
Figure 3.5 **Geodetic Coordinates** $\{\phi,\lambda\}$

GISC-3325

16 February 2010

The Sphere

- Sphere can be used as a first approximation to the geoid.
  - Fits well locally
- Globally it is a poor approximation.
  - Difference between equatorial and polar radius: 21km



Ellipsoid

- Rotational or biaxial ellipsoid is a good approximation both locally and globally.
  - Distance between the geoid and best-fitting ellipsoid does not exceed 100 meters.
- Its geometry is relatively simple and so are the computations on the ellipsoidal surface.

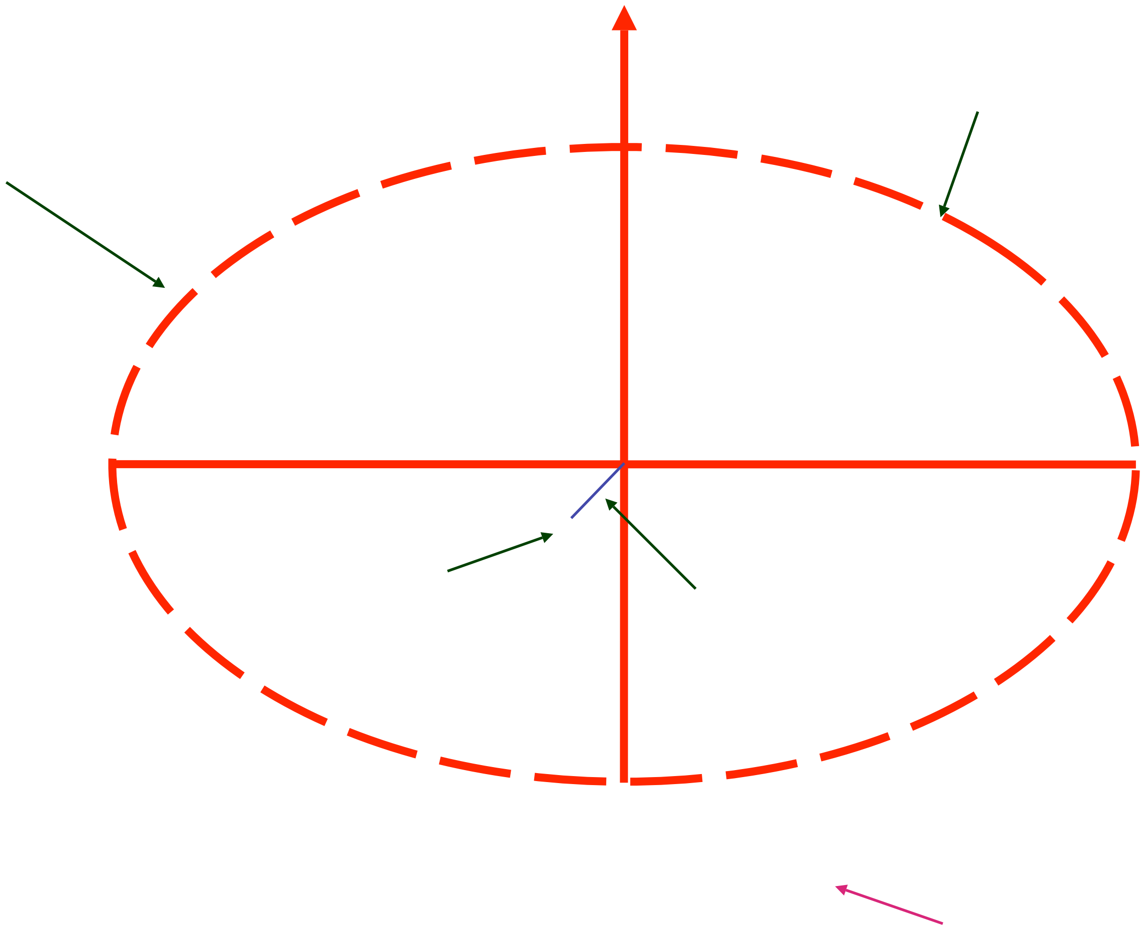
Ellipsoid Characteristics

- All parallels are still circles on the ellipse
  - Each is successively smaller moving to poles.
- We can calculate lengths and areas with additional difficulty due to the convergence of meridians.
- Most computations do not have “closed forms” - they require iteration.

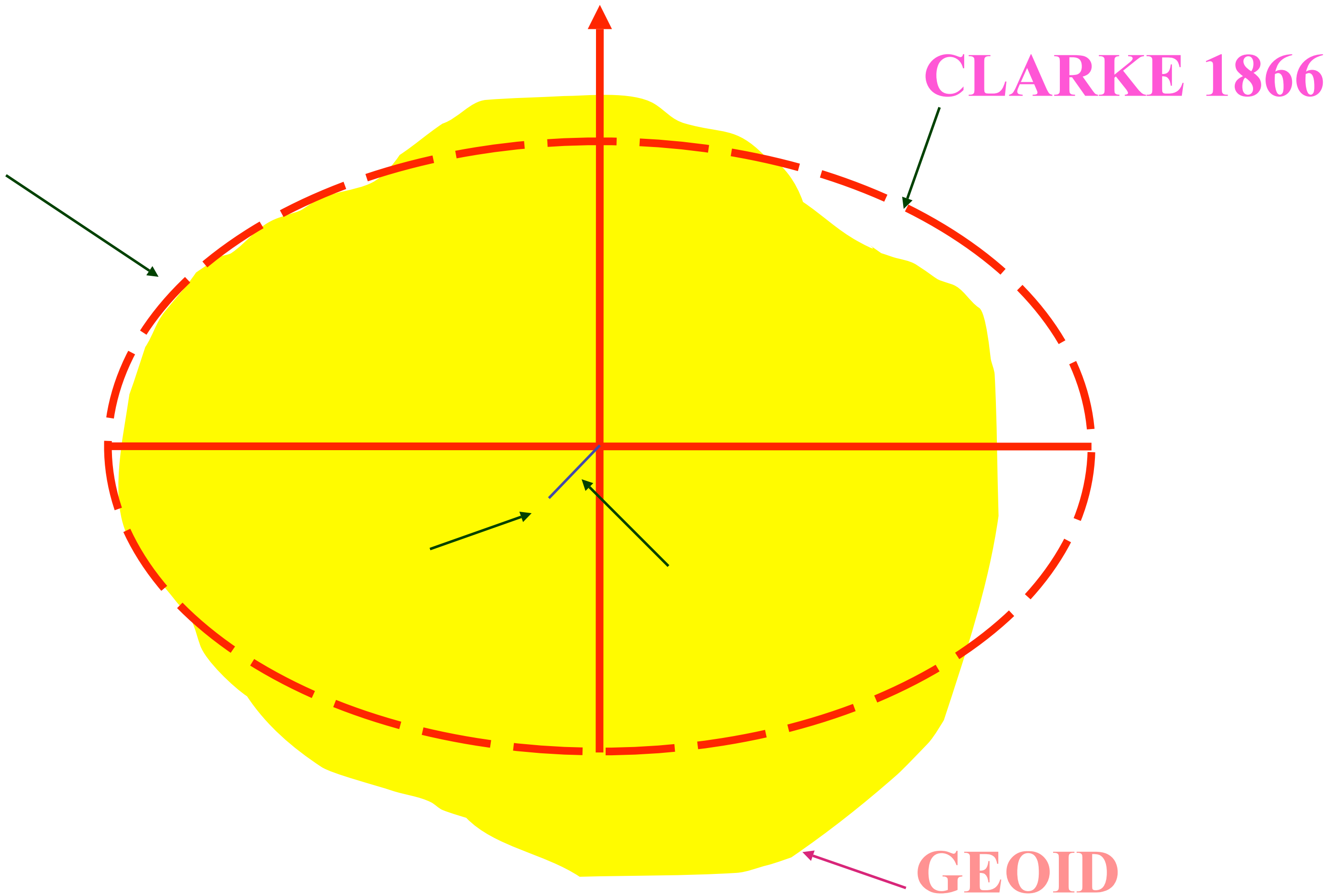
Complications

- Ellipsoid does not possess a constant radius of curvature.
- Radii of curvature are dependent on latitude only.
- Two radii are of interest
  - Meridian
  - Prime Vertical

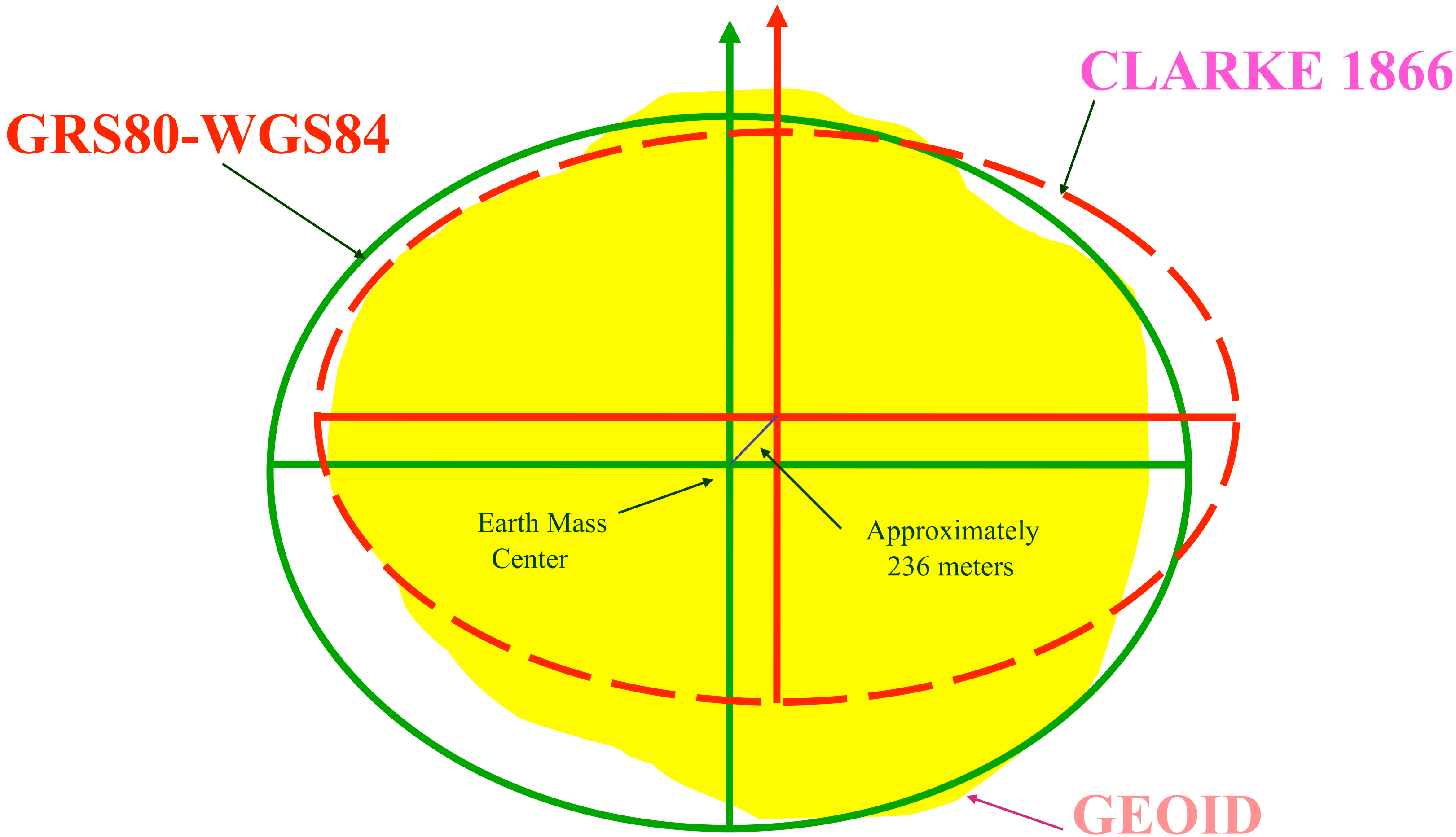
THE GEOID AND TWO ELLIPSOIDS



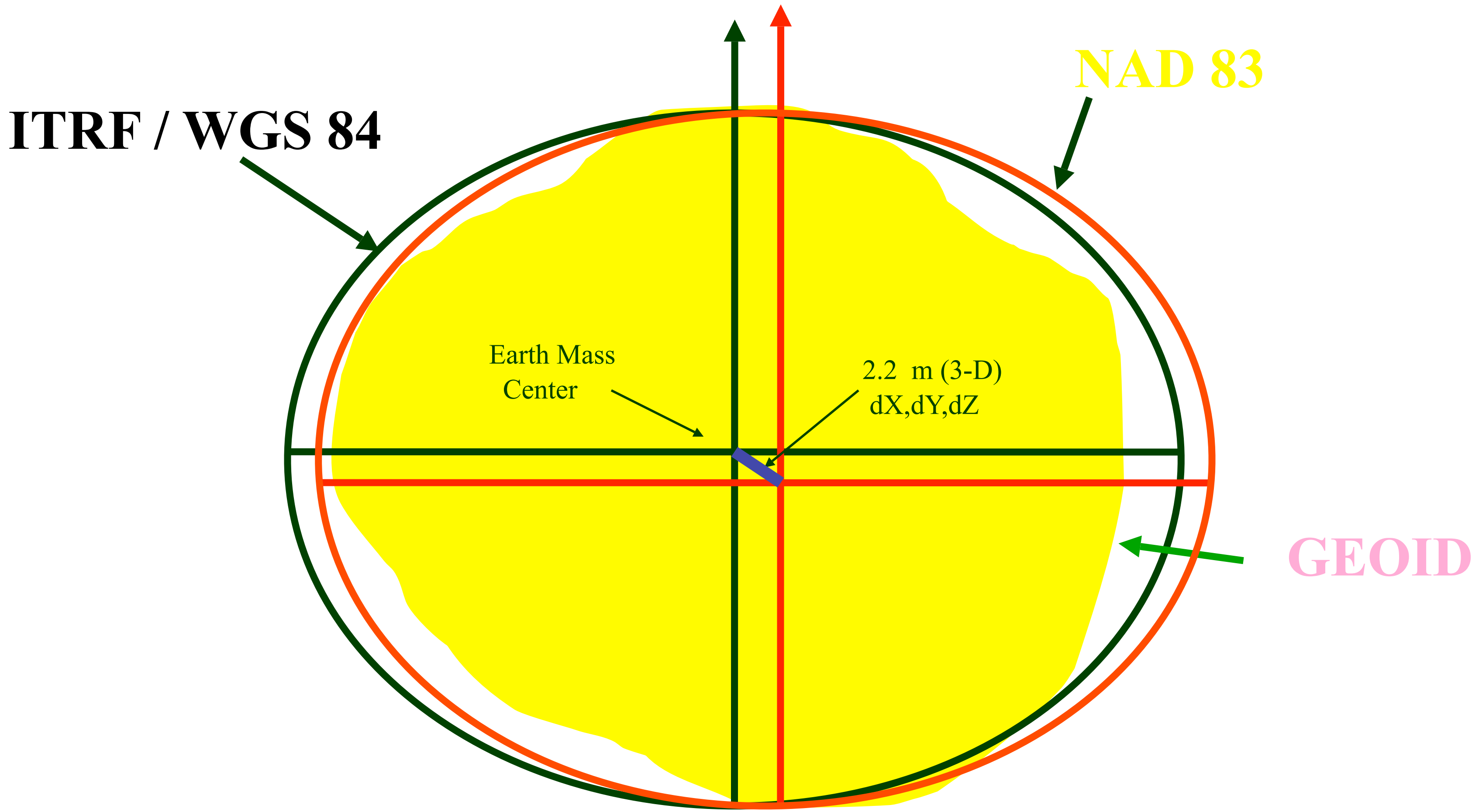
THE GEOID AND TWO ELLIPSOIDS

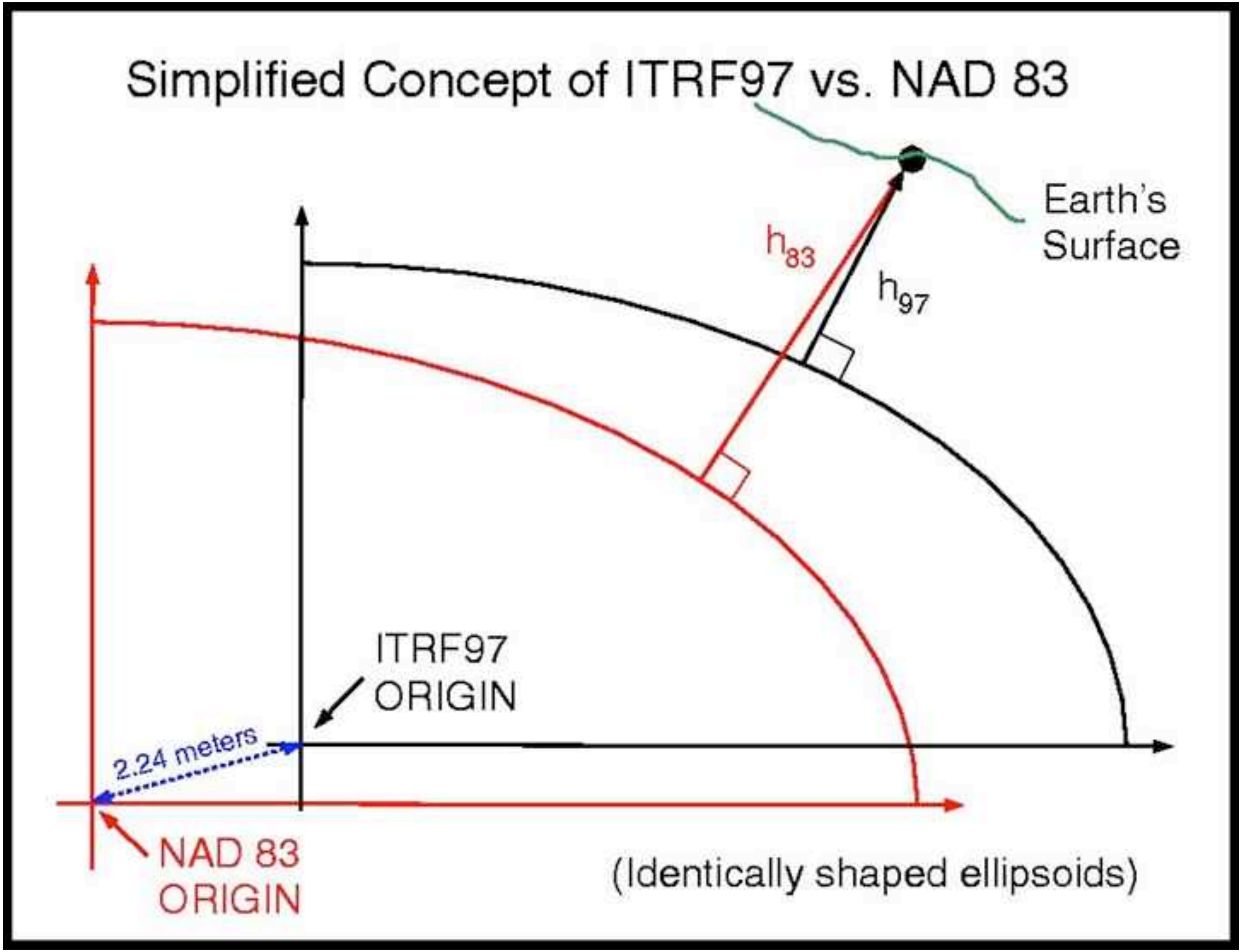


THE GEOID AND TWO ELLIPSOIDS



NAD 83 and ITRF / WGS 84

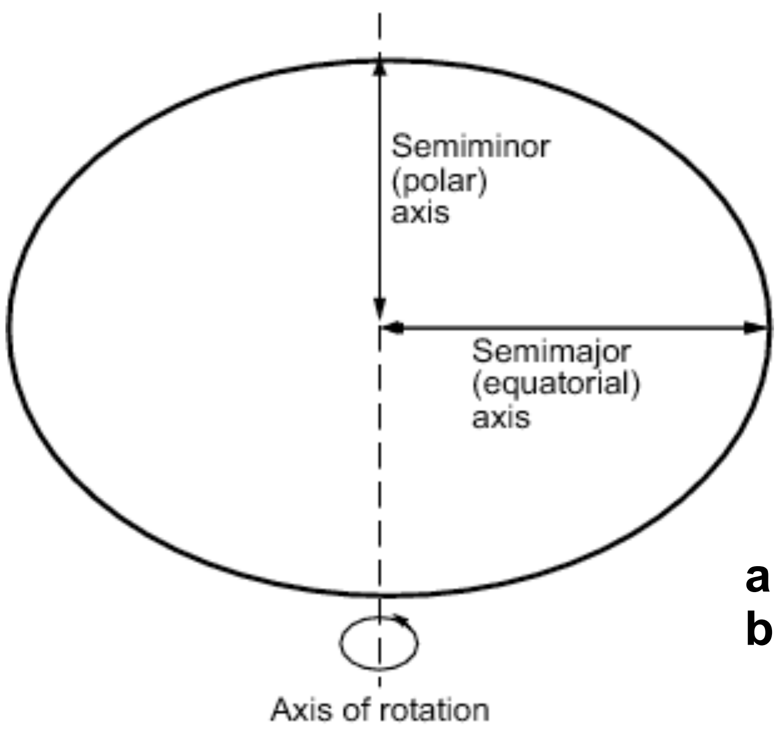




Simplified Concept of ITRF97 vs. NAD 83
Earth's Surface
h_{83}
h_{97}
ITRF97 ORIGIN
2.24 meters
NAD 83 ORIGIN
(Identically shaped ellipsoids)

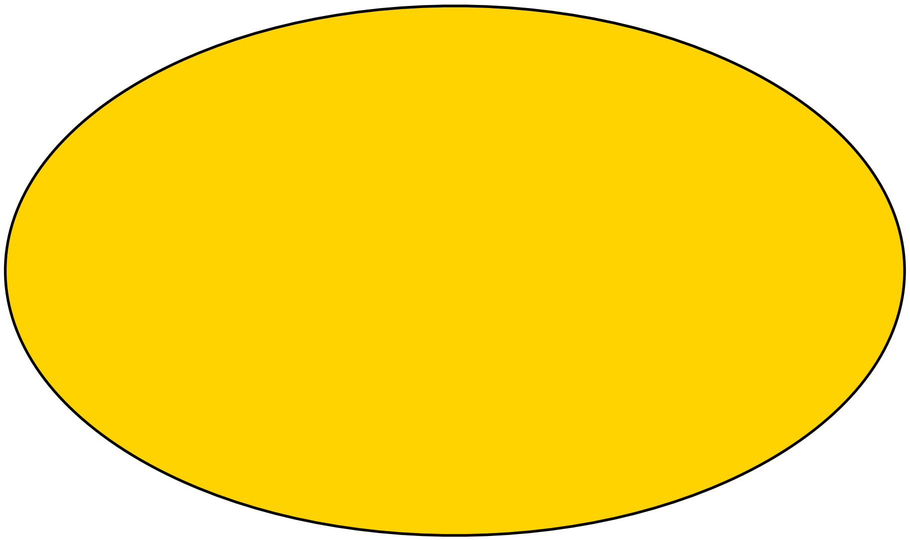
Ellipsoid - geometric

- Can be defined purely in terms of its geometric parameters: two semi-axes a and b or one semi-axis and the flattening.

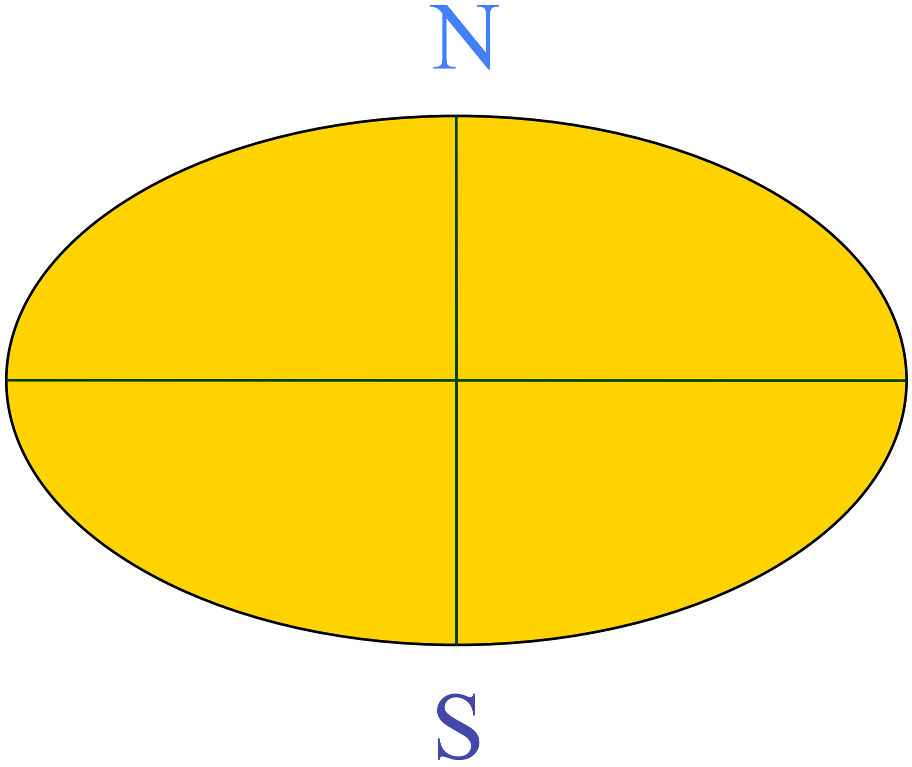


a = semi-major axis
b = semi-minor axis

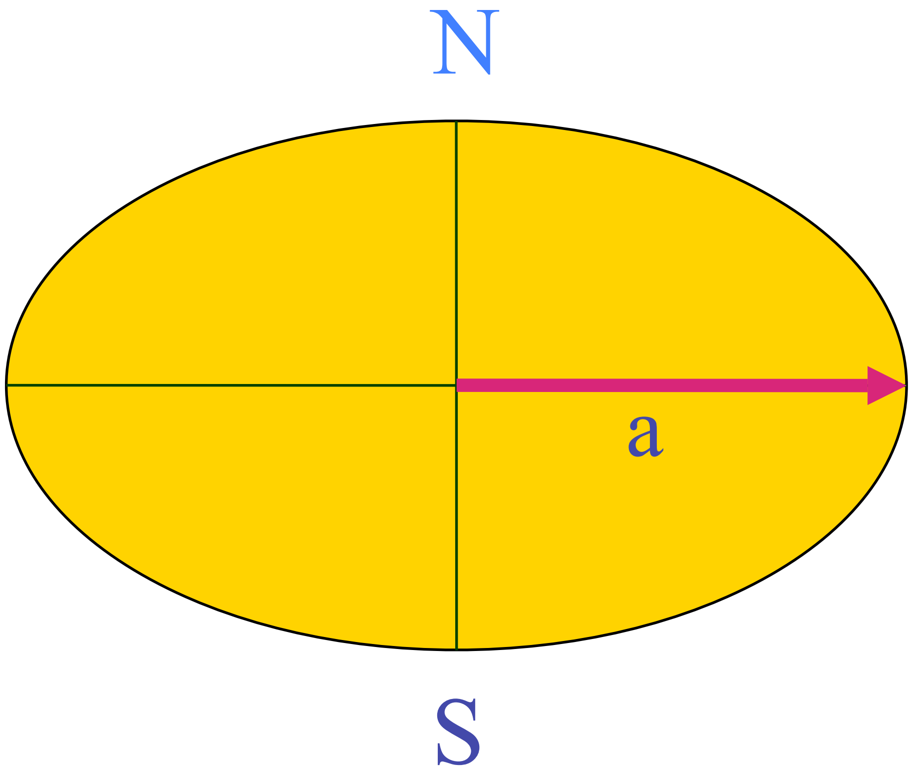
THE ELLIPSOID
MATHEMATICAL MODEL OF THE EARTH



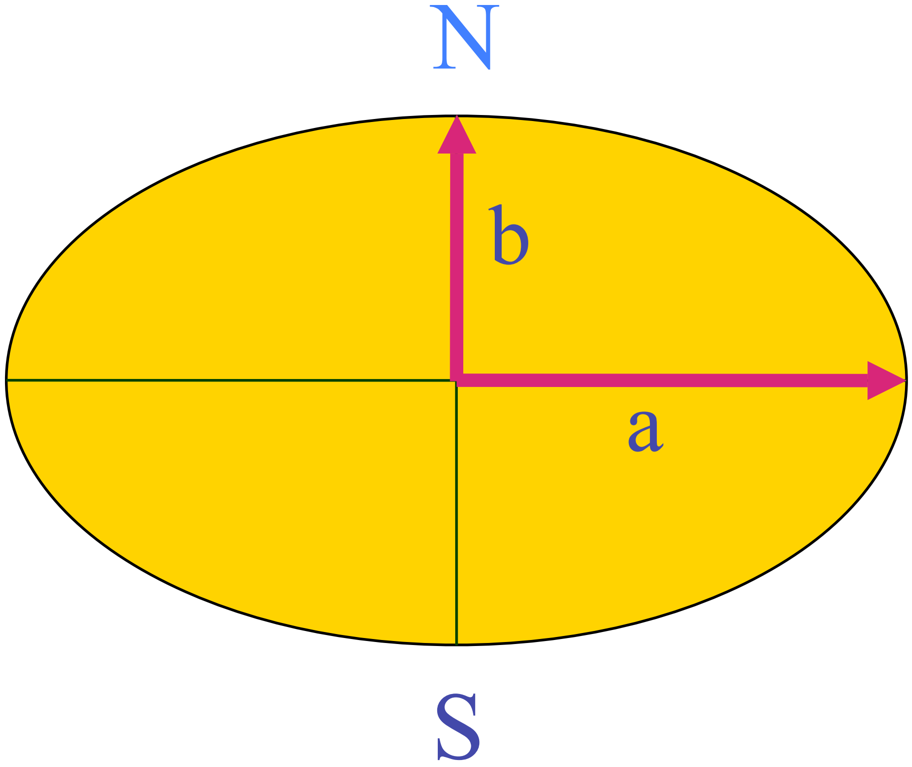
THE ELLIPSOID
MATHEMATICAL MODEL OF THE EARTH



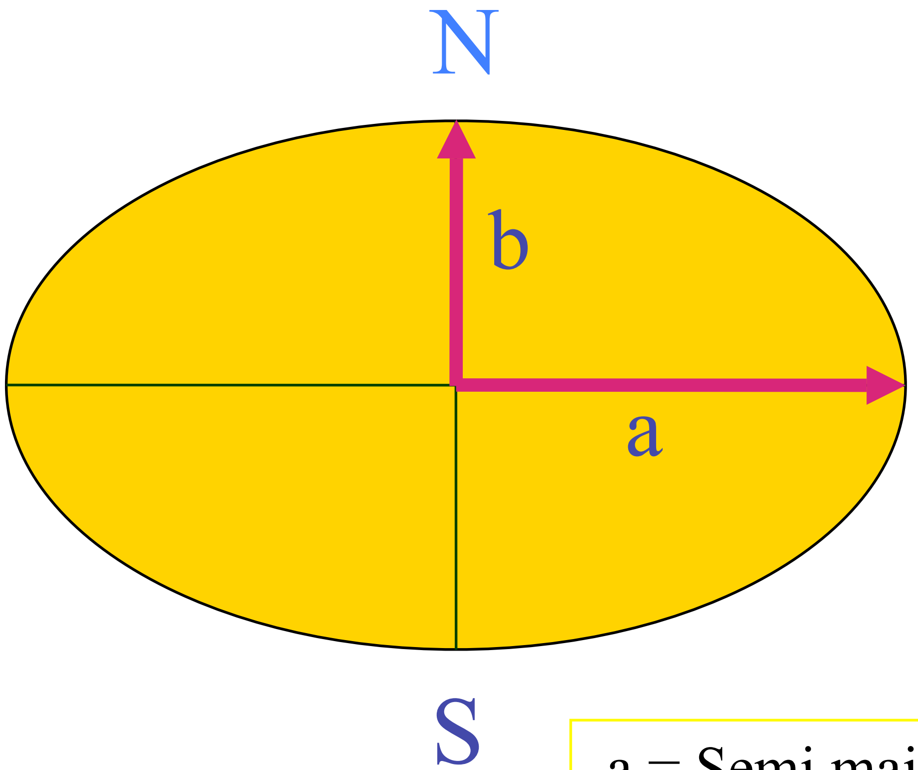
THE ELLIPSOID
MATHEMATICAL MODEL OF THE EARTH



THE ELLIPSOID
MATHEMATICAL MODEL OF THE EARTH



THE ELLIPSOID
MATHEMATICAL MODEL OF THE EARTH



a = Semi major axis

b = Semi minor axis

$f = \frac{a-b}{a}$ = Flattening

Geometric Parameters

- a = semi-major axis length
- b = semi-minor axis length
- f = flattening = (a-b)/a
- e = first eccentricity = $\sqrt{(a^2-b^2)/a^2}$ alternately $e^2 = (a^2-b^2)/a^2$
- e’ = second eccentricity = $\sqrt{(a^2-b^2)/b^2}$

Ellipsoid - Physical

- Takes into account the gravity potential of the surface and external to it.
- Additional parameters used to define the reference gravity potential (U):
  - Mass of the Earth
  - Rotation rate
- Resulting gravity on the surface of the Earth is called normal gravity
- GRS 80 defines geometry and physical.

Ellipsoidal Surface as Equipotential Surface

- Originates from reference surface potential U.
- Ellipsoid height (h) can be expressed as a function of U and N (geoid height) as a function of the potential difference.
- Orthometric height (H) are a function of the gravity potential W(x,y,z)
- The disturbing potential (T) = W – U
  - N = f(W-U) = f(T)

Geodetic Reference System 1980 (GRS80)

adopted by the International Association of Geodesy (IAG) during the General Assembly 1979. Principal parameters are:

parameter	symbol	value
defining constants		
equatorial radius of the Earth	a	6378137 m
geocentric gravitational constant (including the atmosphere)	GM	$3986005 \cdot 10^{8}\ m^{3}s^{-2}$
dynamical form factor (excluding permanent tides)	J_2	$108263 \cdot 10^{-8}$
angular velocity of the Earth	w	$7292115 \cdot 10^{-11}$ rad s^{-1}
derived geometrical parameters		
semiminor axis (polar radius)	b	6356752.3141 m
first excentricity	e^2	0.00669438002290
flattening	f	1 : 298.257222101
mean radius	R_1	6371008.7714 m
radius of sphere with same surface	R_2	6371007.1810 m
radius of sphere with same volume	R_3	6371000.7900 m
derived physical parameters		

Ellipsoid Models in U.S.

- Clarke Spheroid of 1866
  - a = 6,378,206.4 m
  - b = 6,356,583.8 m
- Geodetic Reference System of 1980
  - a = 6,378,137.0 m
  - 1/f = 298.257222101
- Where a = semi-major, b = semi-minor and 1/f is inverse of flattening.

Selected Reference Ellipsoids

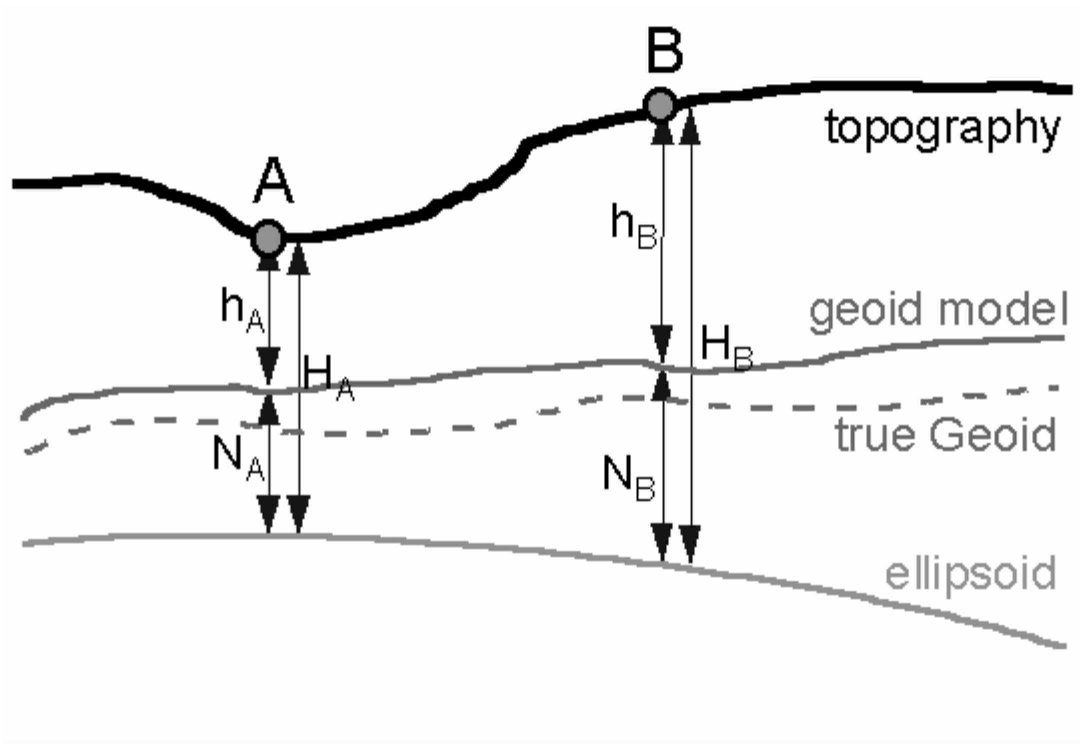
Ellipse	Semi-Major Axis (meters)	1/Flattening
Airy 1830	6377563.396	299.3249646
Bessel 1841	6377397.155	299.1528128
Clarke 1866	6378206.4	294.9786982
Clarke 1880	6378249.145	293.465
Everest 1830	6377276.345	300.8017
Fischer 1960 (Mercury)	6378166.0	298.3
Fischer 1968	6378150.0	298.3
G R S 1967	6378160.0	298.247167427
G R S 1975	6378140.0	298.257
G R S 1980	6378137.0	298.257222101
Hough 1956	6378270.0	297.0
International	6378388.0	297.0
Krassovsky 1940	6378245.0	298.3
South American 1969	6378160.0	298.25
WGS 60	6378165.0	298.3
WGS 66	6378145.0	298.25
WGS 72	6378135.0	298.26
WGS 84	6378137.0	298.257223563

Peter H. Dana 9/1/94

Best-fitting ellipsoid

- An ellipsoid satisfying the condition that the deviations between the geoid and ellipsoid (in a global sense) are minimized.
  - Computed by least squares methods.
- Distance **N** between the geoid and best fitting ellipsoid is called geoidal undulation and can be computed from: $\mathbf{N \approx h - H}$
  - **Geoidal undulation ≈ ellipsoid height – orthometric height**

$N \approx h - H$



Where h is geometric and H is physical

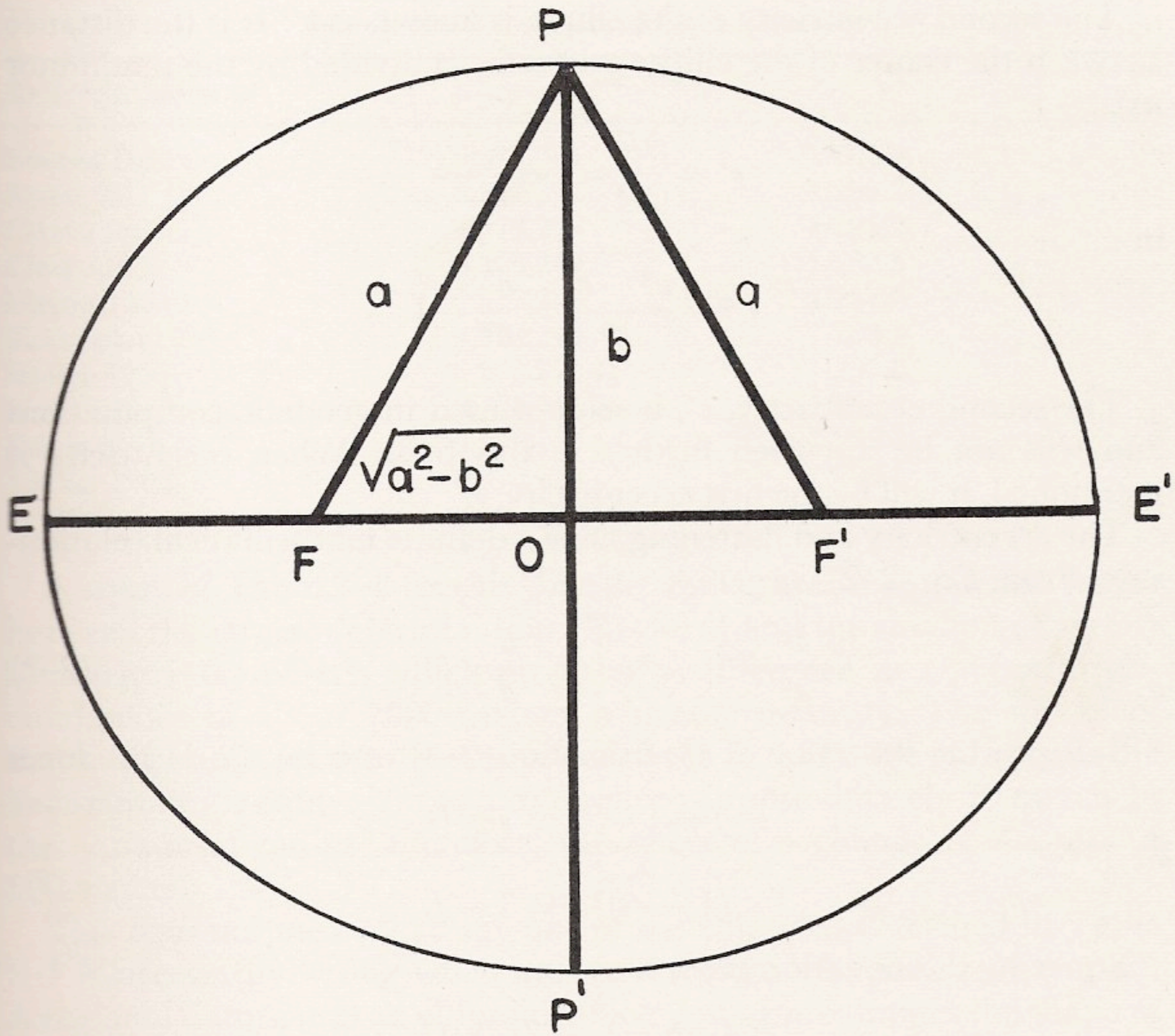


Figure 2–3. Some basic properties of the ellipse.

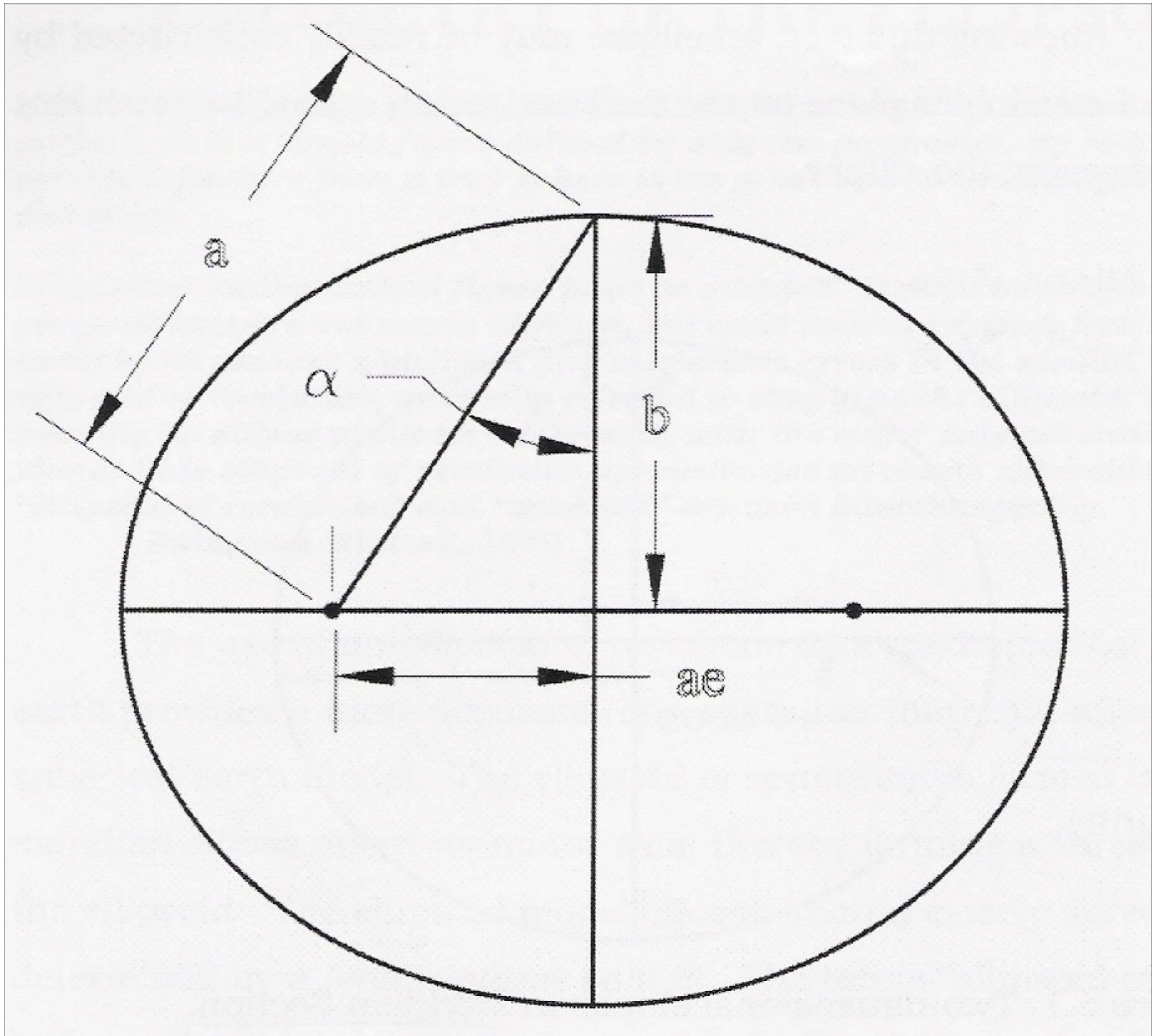


Figure 5.2 Angular Eccentricity.

Distance $ae^2 = a^2 - b^2$

$\alpha = \tan^{-1}(ae/b)$

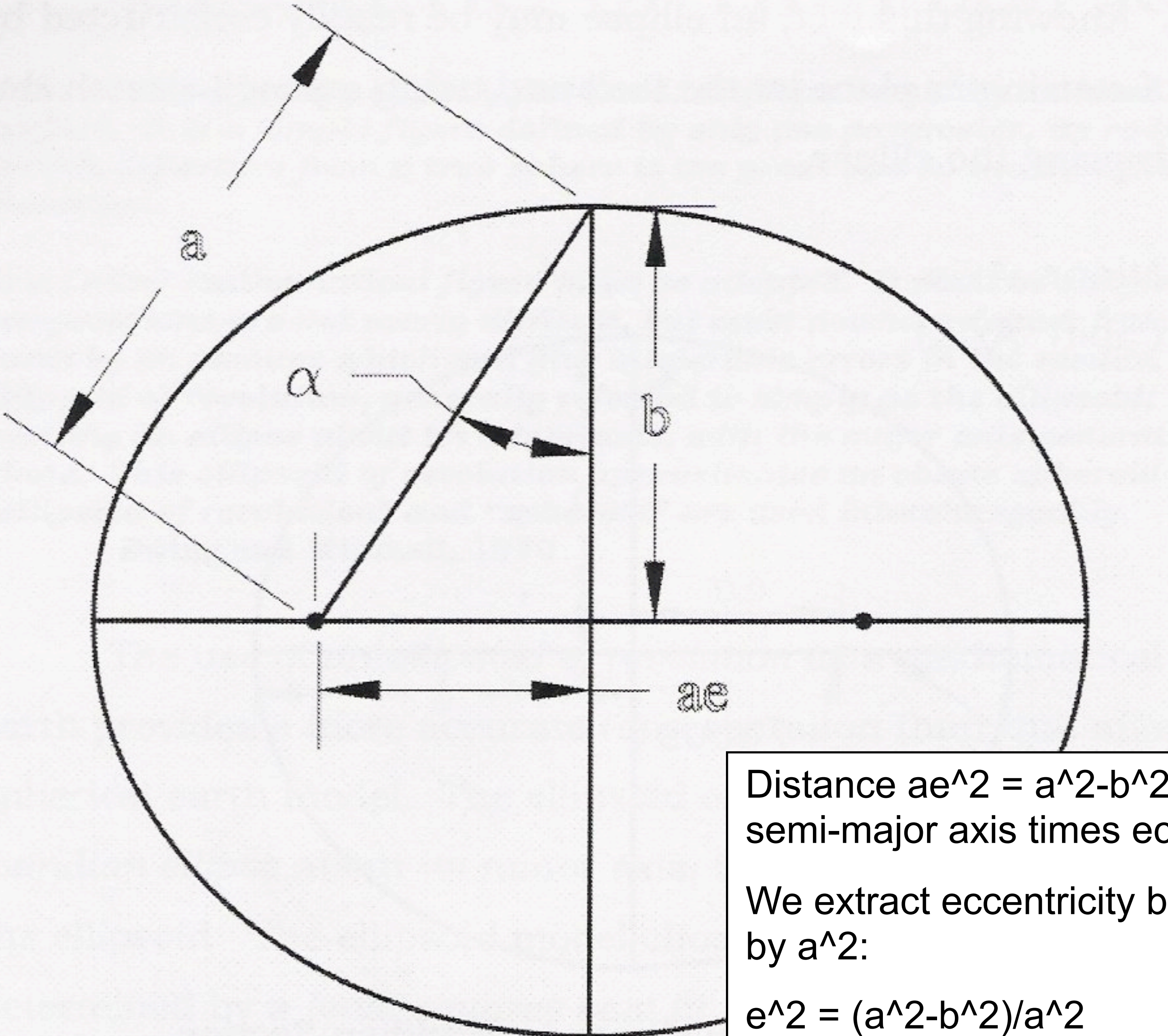


Figure 5.2 Angular Eccentricity.

Distance ae^2 = a^2-b^2 where (ae) is the semi-major axis times eccentricity (e)

We extract eccentricity by dividing both sides by a^2:

e^2 = (a^2-b^2)/a^2

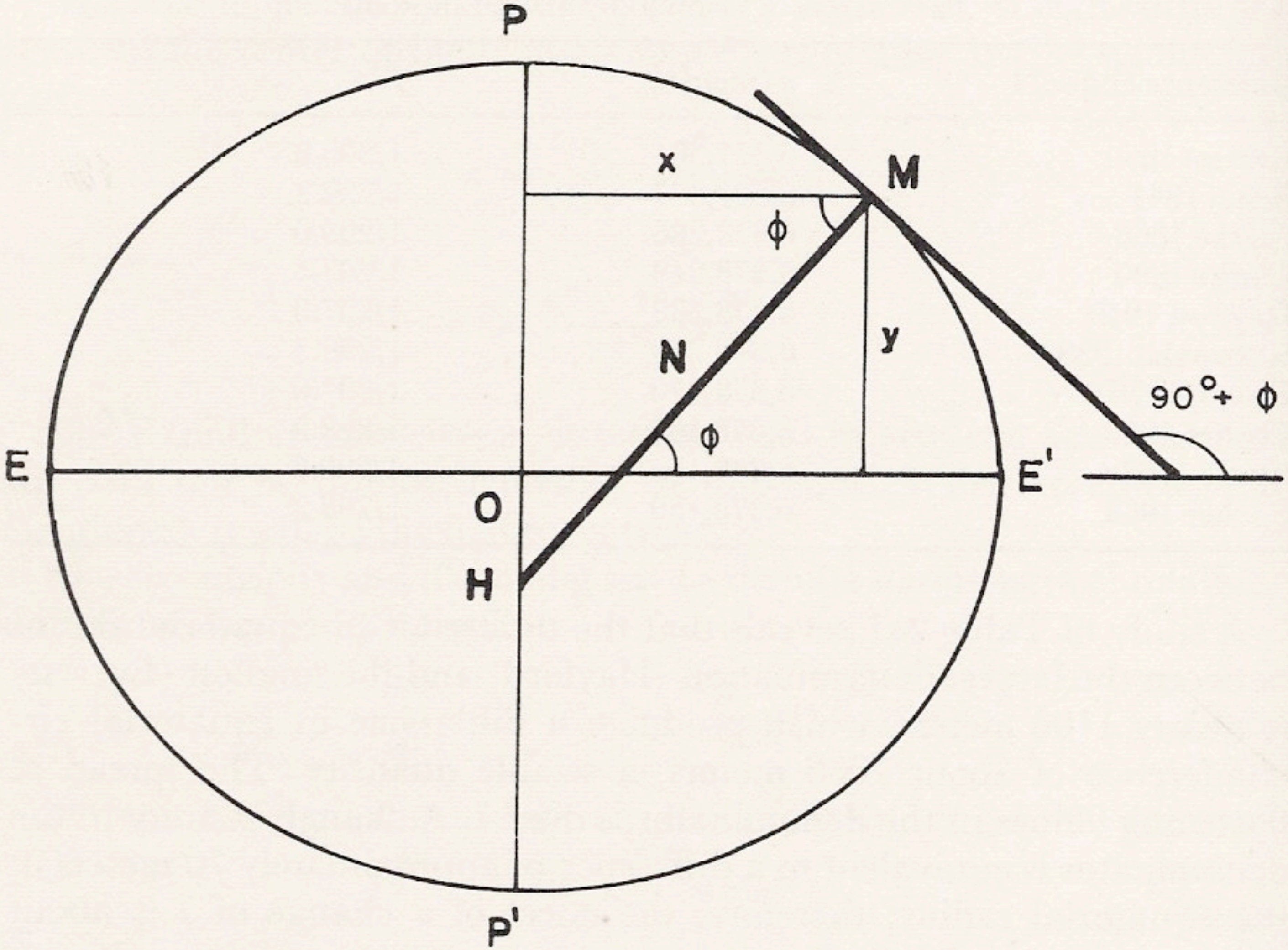


Figure 2–4. Additional properties of the ellipse.

Three Latitudes

- Geodetic latitude included angle formed by the intersection of the ellipsoid normal with the major (equatorial) axis.
- Geocentric latitude included angle formed by the intersection of the line extending from the point on the ellipse to the origin of axes.
- Parametric (reduced) latitude is the included angle formed by the intersection of a line extending from the projection of a point on the ellipse onto a concentric circle with radius = a

Geodetic latitude

Geocentric latitude

Parametric latitude

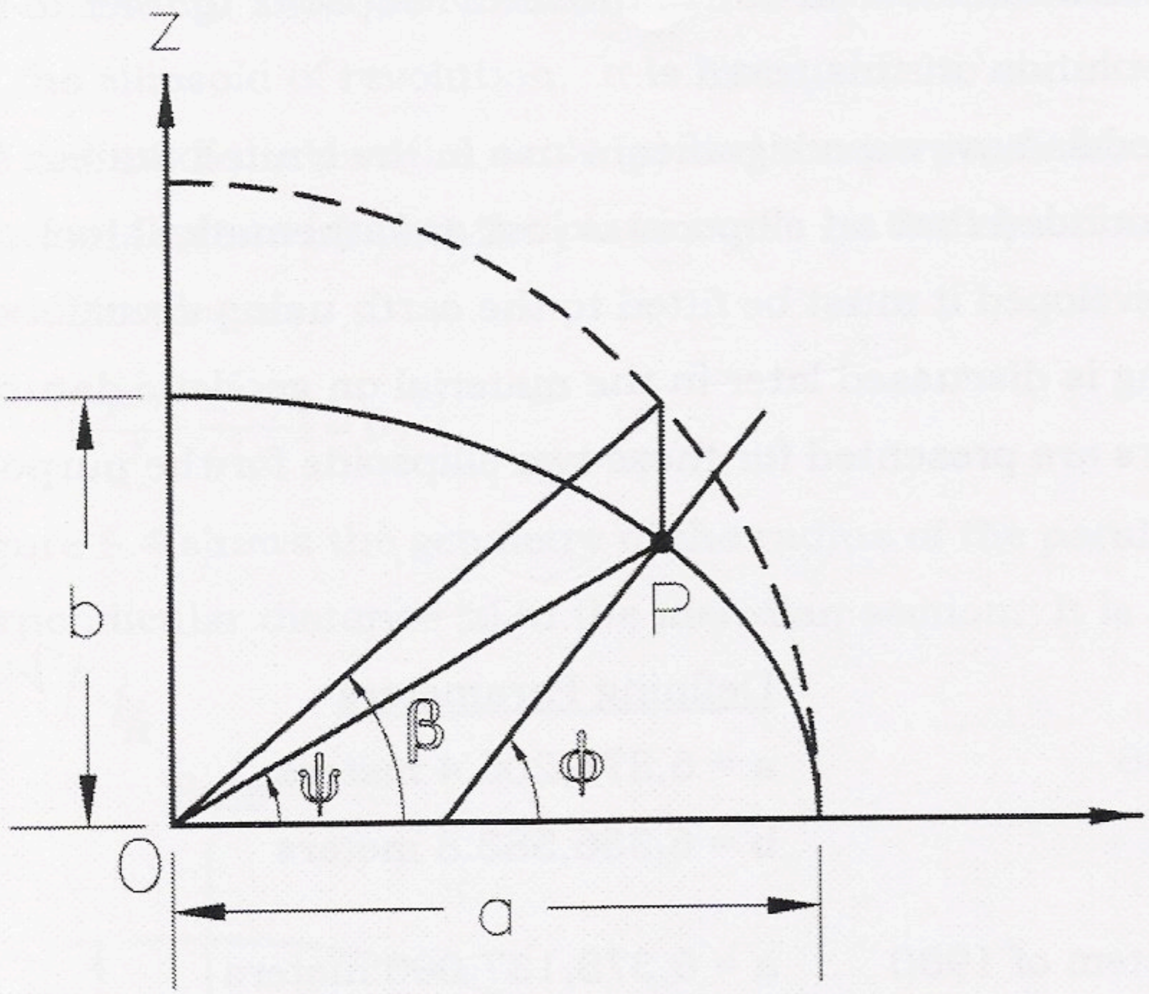
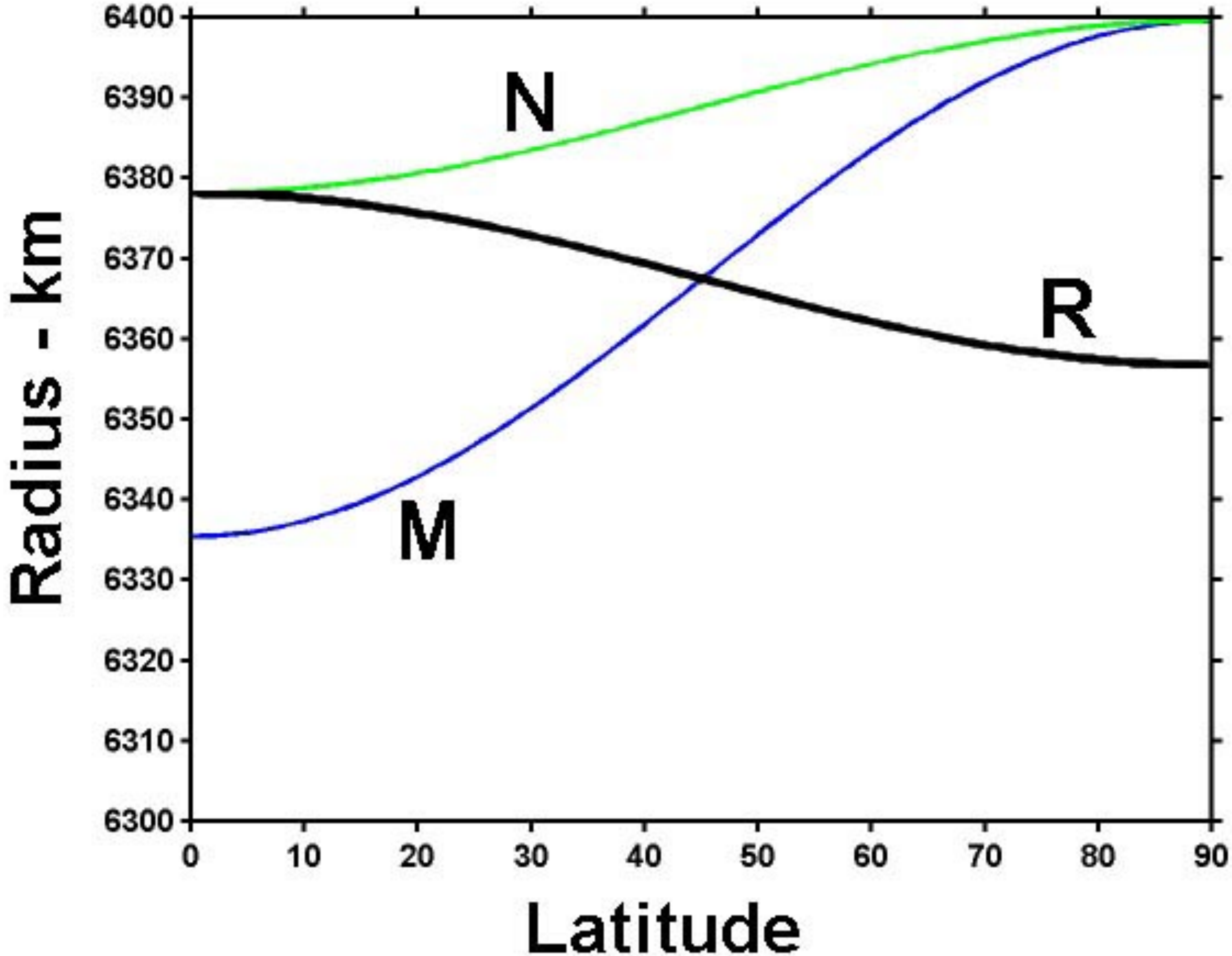
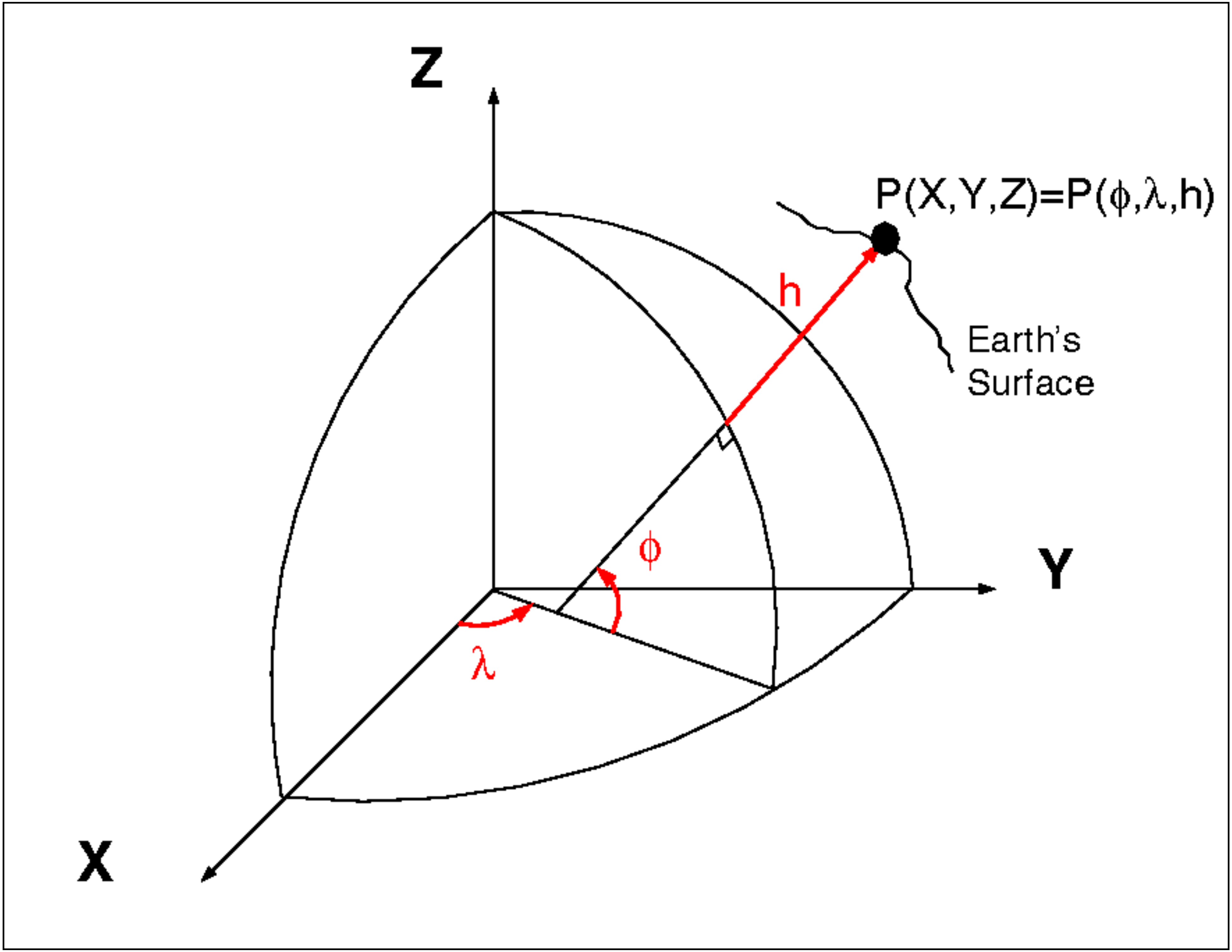


Figure 5.3 The three latitudes.

Unlike the sphere, the ellipsoid does not possess a constant radius of curvature.



Radius - km
6400
6390
6380
6370
6360
6350
6340
6330
6320
6310
6300
N
R
M
0
10
20
30
40
50
60
70
80
90
Latitude


Z
P(X,Y,Z)=P(φ,λ,h)
h
Earth's
Surface
φ
Y
λ
X

Geocentric relationship to XYZ

- One of the advantages of geocentric angles is that the relationship to XYZ is easy. R is taken to be radius of the sphere and H the height above this radius

$$\phi_c = \tan^{-1}(Z/\sqrt{X^2+Y^2})$$
$$\lambda_c = \tan^{-1}(Y/X)$$
$$R + H_c = \sqrt{X^2+Y^2+Z^2}$$

$$X = (R + H_c)\cos\phi_c \cos\lambda_c$$
$$Y = (R + H_c)\cos\phi_c \sin\lambda_c$$
$$Z = (R + H_c)\sin\phi_c$$

$$(\lambda, \phi, h_e)_{a,f} = g(x, y, z)$$

$$h_e = \frac{\sqrt{x^2 + y^2}}{\cos\phi} - N$$

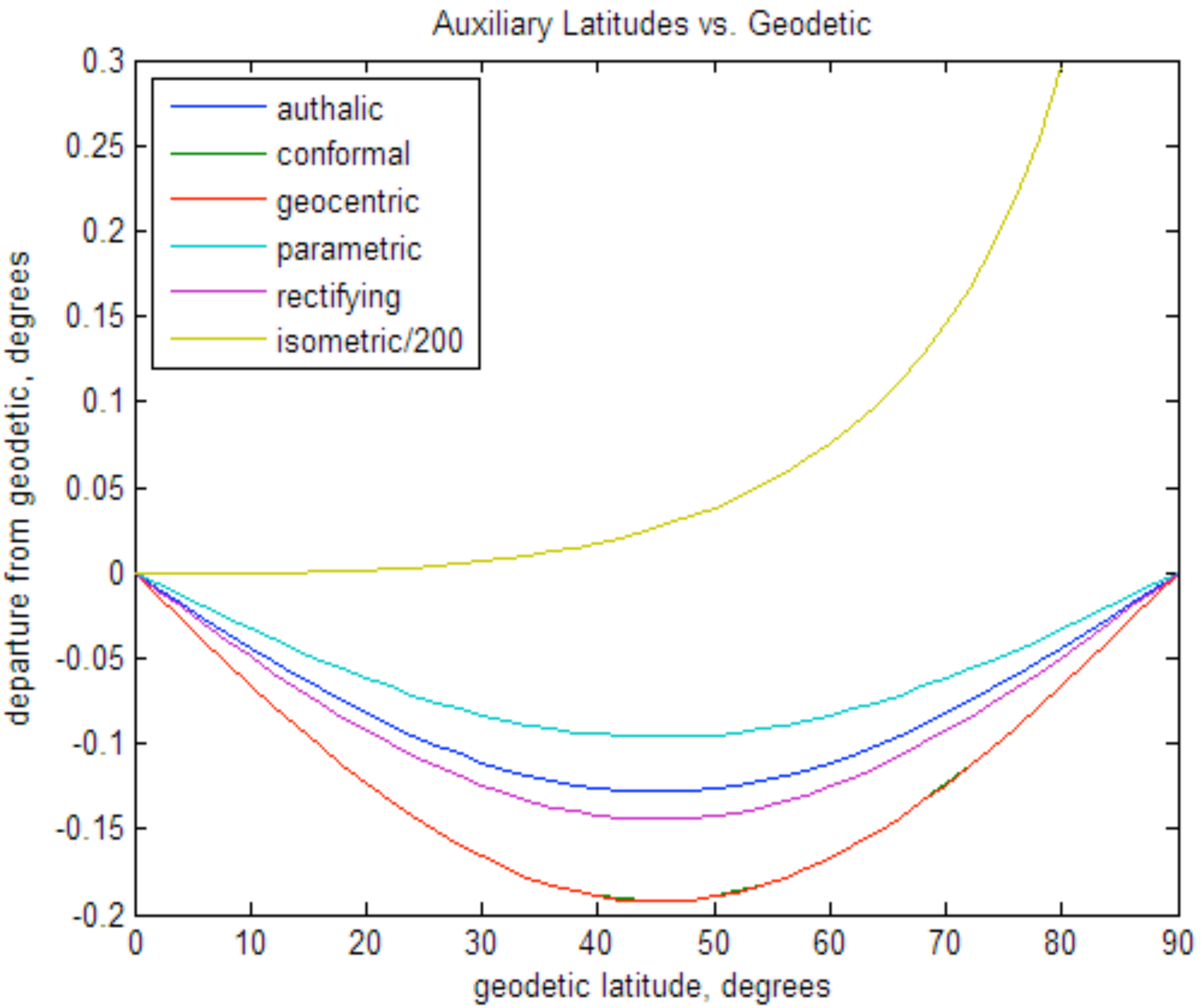
$$\phi = \arctan\left\{\frac{z}{\sqrt{x^2 + y^2}}\left[1 - e^2\left(\frac{N}{N + h_e}\right)\right]^{-1}\right\}$$

$$\lambda = \arctan\left(\frac{y}{x}\right)$$

$$N = \frac{a \cos\phi}{\cos\phi \left(1 - e^2 \sin^2 \phi\right)^{1/2}},$$

$$\therefore \quad N = \frac{a}{\left(1 - e^2 \sin^2 \phi\right)^{1/2}}.$$

Comparison of latitudes



Converting latitudes from geodetic

- Parametric latitude = arctan($\sqrt{}$(1-e^2)*tan(lat))
- Geocentric latitude = arctan((1-e^2)*tan(lat))

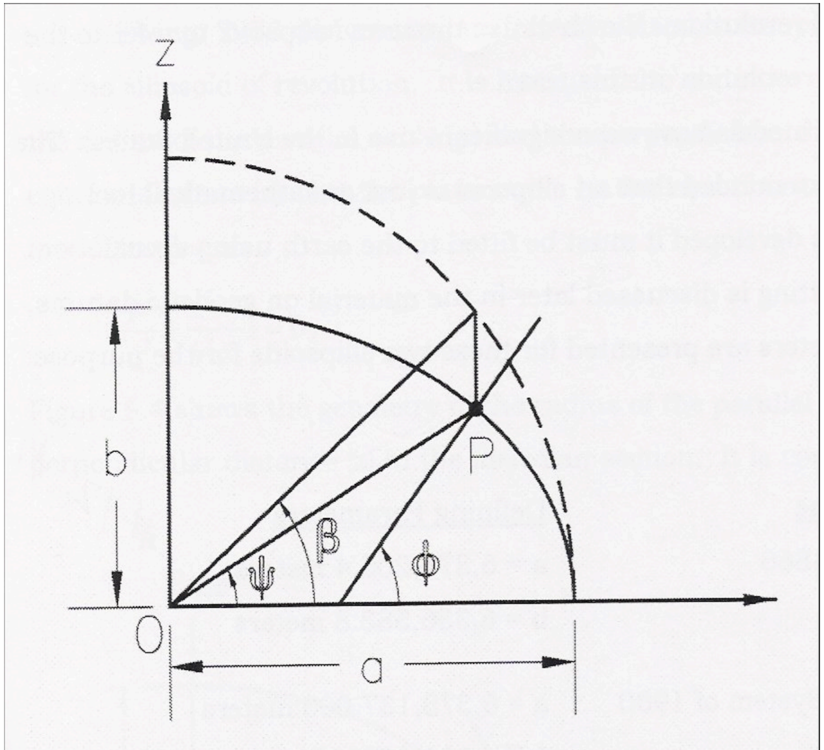


Figure 5.3 The three latitudes.

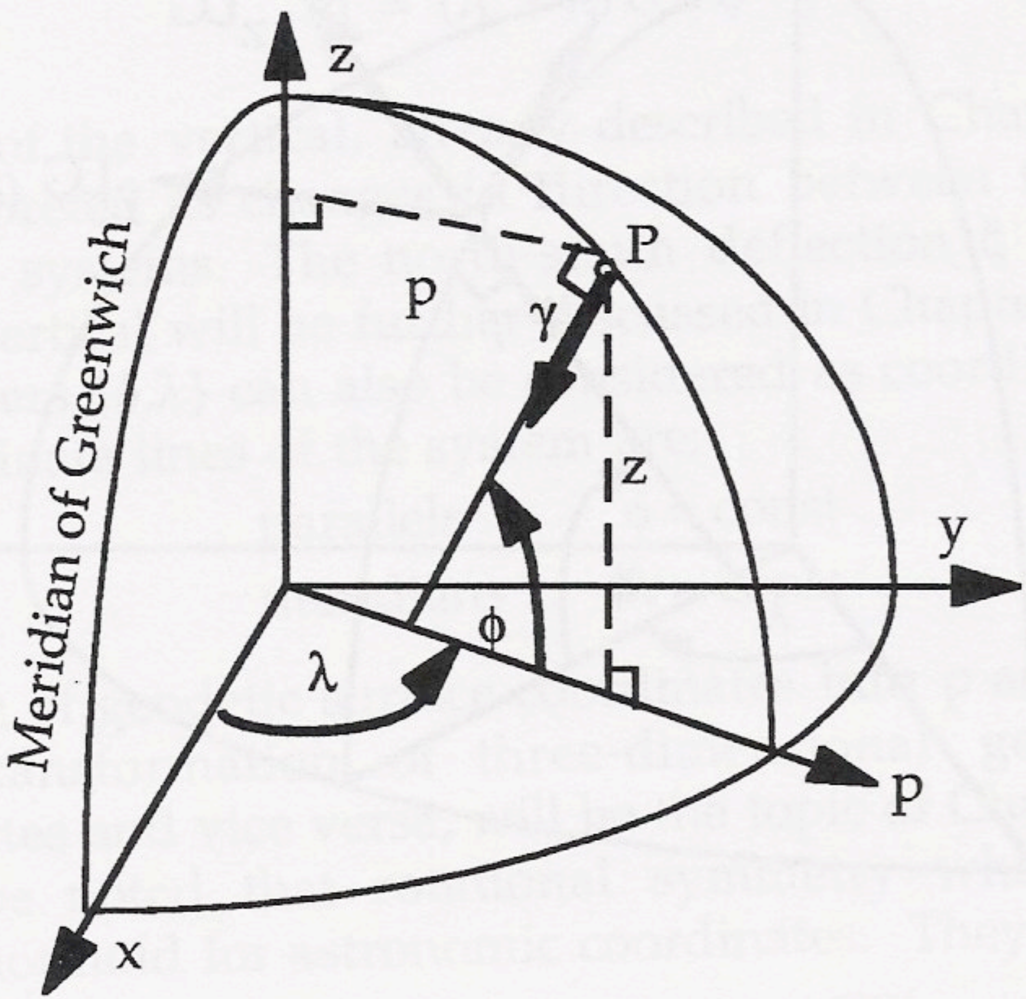


Figure 3.5 Geodetic Coordinates {φ,λ}

$$p = \frac{a\cos\phi}{\left(1 - e^2 \sin^2 \phi\right)^{1/2}}$$

$$z = \frac{a\left(1 - e^2\right)\sin\phi}{\left(1 - e^2 \sin^2 \phi\right)^{1/2}}$$

Radius of Curvature in Prime Vertical

$$N = \frac{a\cos\phi}{\cos\phi\left(1-e^2\sin^2\phi\right)^{1/2}},$$

$$\therefore \quad N = \frac{a}{\left(1-e^2\sin^2\phi\right)^{1/2}}.$$

- N extends from the minor axis to the ellipsoid surface.
- N >= M
- It is contained in a special normal section that is oriented 90 or 270 degrees to the meridian.

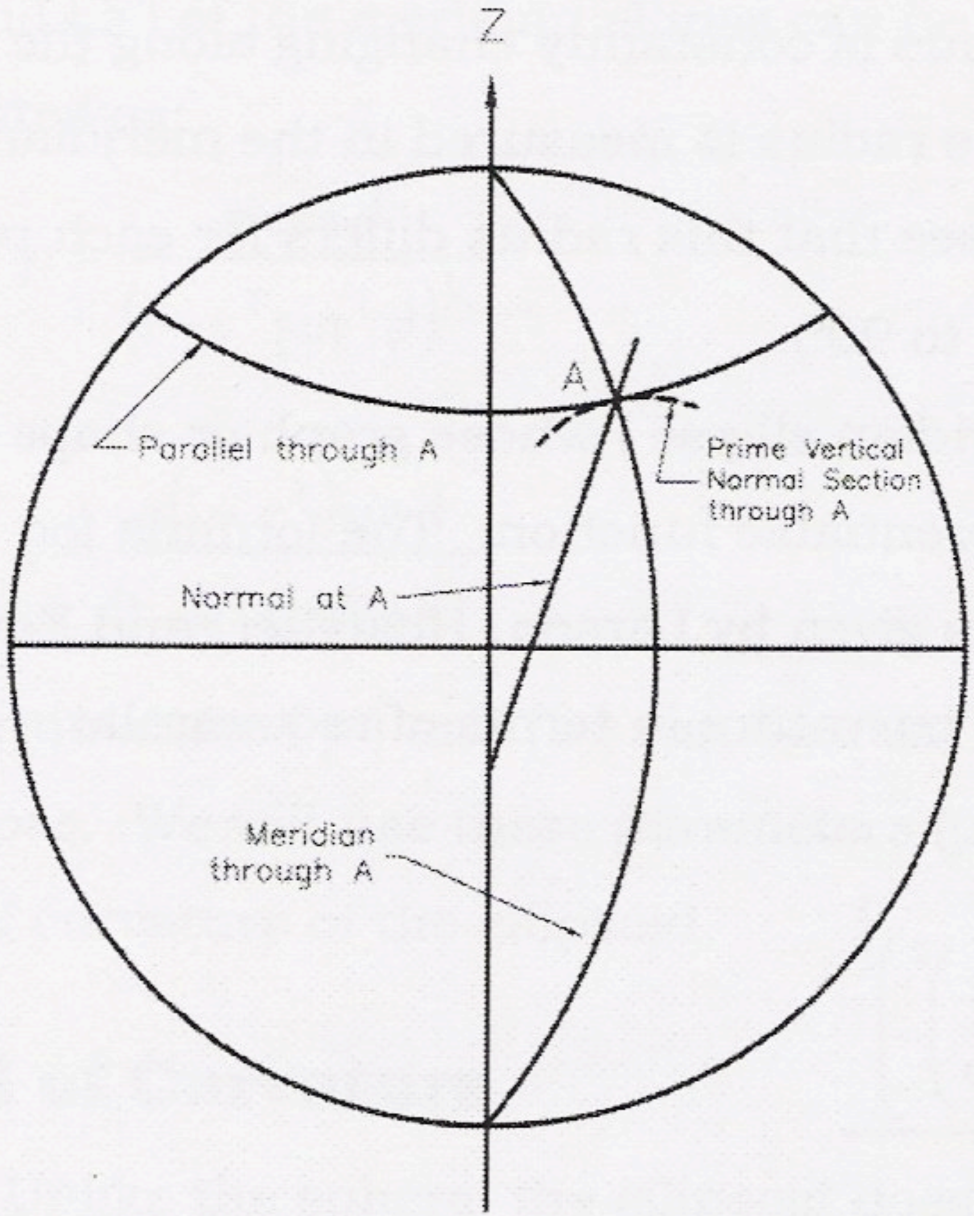


Figure 5.6 Radius of Curvature in the Prime Vertical.

$$\frac{p^2}{a^2}+\frac{z^2}{b^2}-1=0.$$

Figure 5.4 shows the geometry of the radius of the parallel (p) and the perpendicular distance (z) in the meridian section.

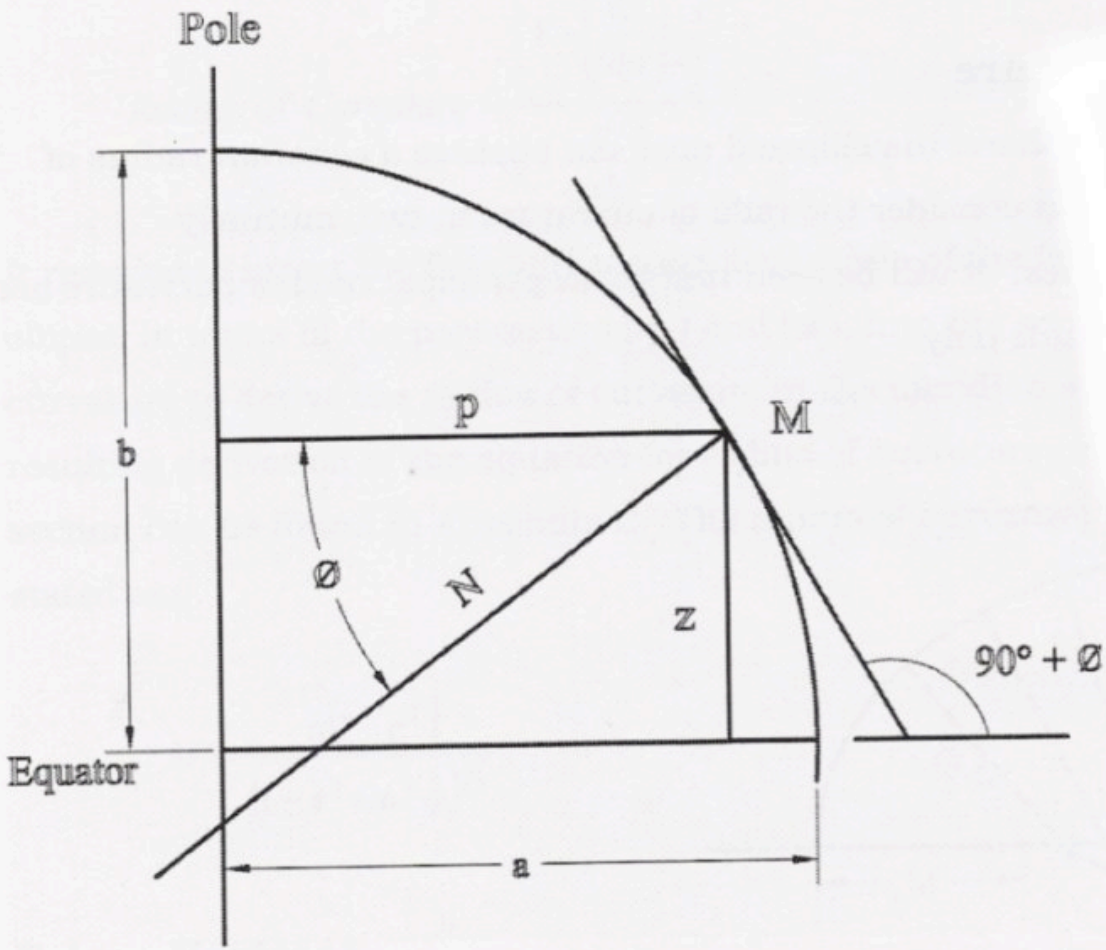


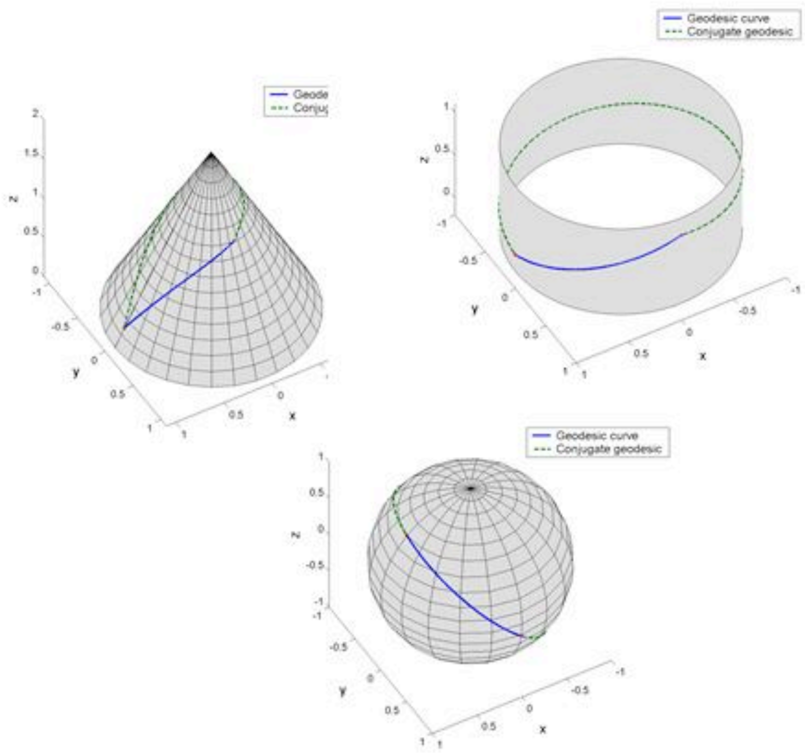
Figure 5.4 Meridian Section of Ellipsoid of Revolution.

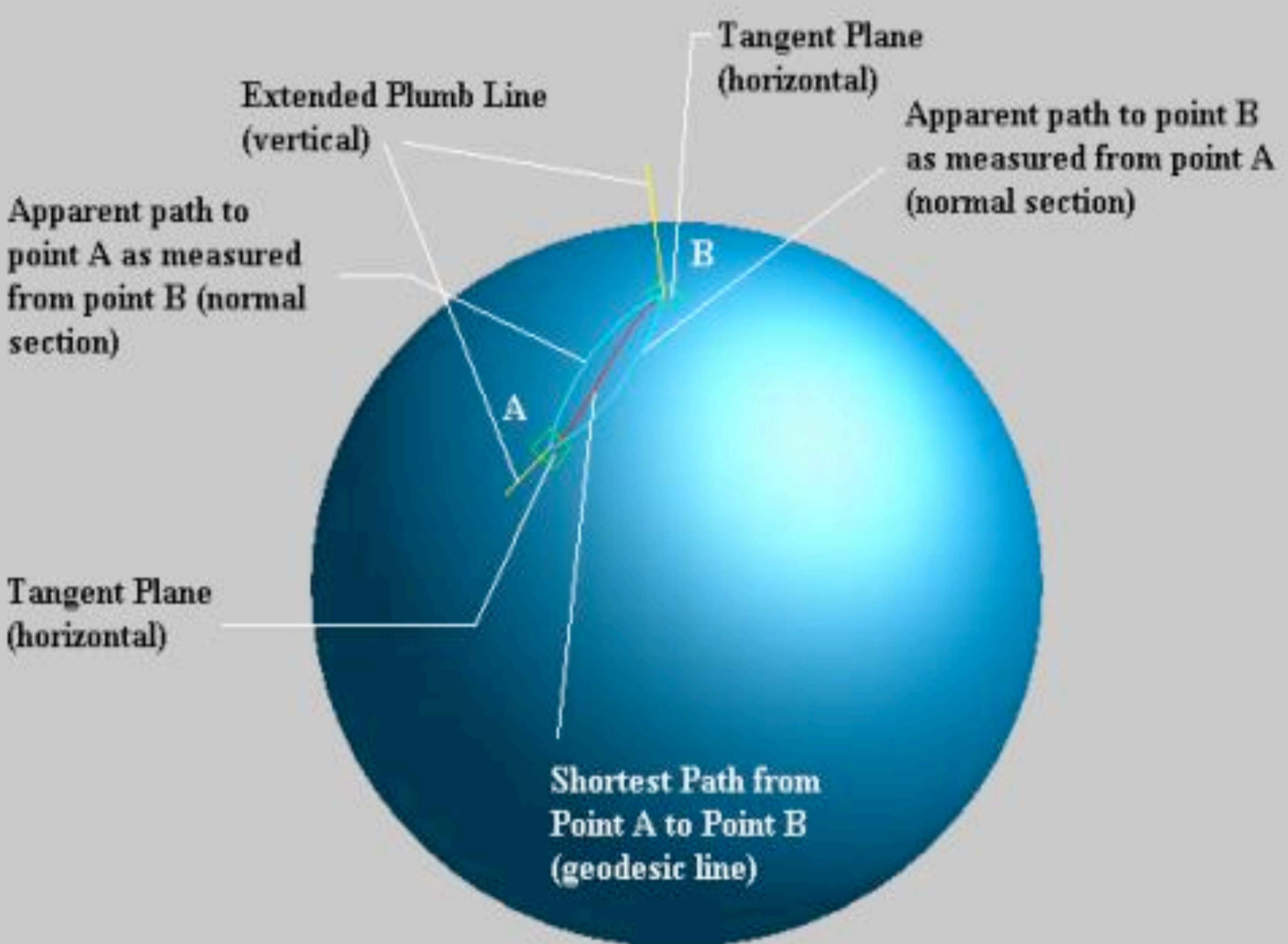
Geodesic

- Analogous to the great circle on the sphere in that it represents the shortest distance between two points on the surface of the ellipsoid.
- Term representing the shortest distance between any two points lying on the same surface.
  - On a plane: straight line
  - On a sphere: great circle

The Geodesic

- It is a generic term referring to the shortest surface distance between any two points on that surface.
- Represents the shortest distance between two points on the surface of the ellipsoid.





Geodesic Lines on an Ellipsoid

Direct and Inverse Problems

- Direct solution computes positions of a new point given a known point, geodetic length and azimuth
- Inverse determines length and azimuth between two points.
- There are a number of approaches to solving including Bowring's detailed in text.
- We now have the NGS toolkit

The NGS approach

- NGS programs INVERSE, FORWARD as well as INVERS3D and FORWARD3D are based on equations developed by T. Vincenty. (on NGS toolkit)
- See:“Direct and Inverse Solutions of Geodesics on the Ellipsoid with Application of Nested Equations.”
  - http://www.ngs.noaa.gov/PUBS_LIB/inverse.pdf

Geodetic Perspectives on the USPLS

- Land ordinance of 1785 authorized the U.S. Public Land Survey System
  - The surveyors…shall proceed to divide the said territory into townships of six miles square, by lines running due north and south, and others crossing them at right angles…”
- USPLSS exists in 30 of the 50 states.
- Many rules and regulations are unique to the system.

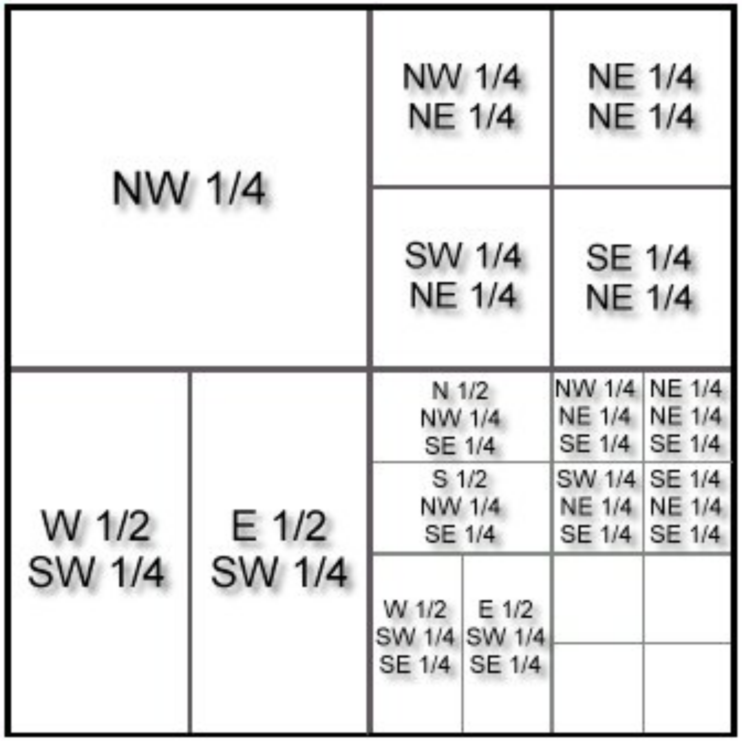
Brief History

- T. Jefferson wanted to have surveys done before land sold in new territories
- Didn’t want to continue with metes and bounds.
- 1785 Ordinance called for townships of 36 one-mile square sections.
- Subsequent legislations established a section as 640 acres.
  - http://www.utexas.edu/depts/grg/huebner/grg312/lect23.html

How System Works

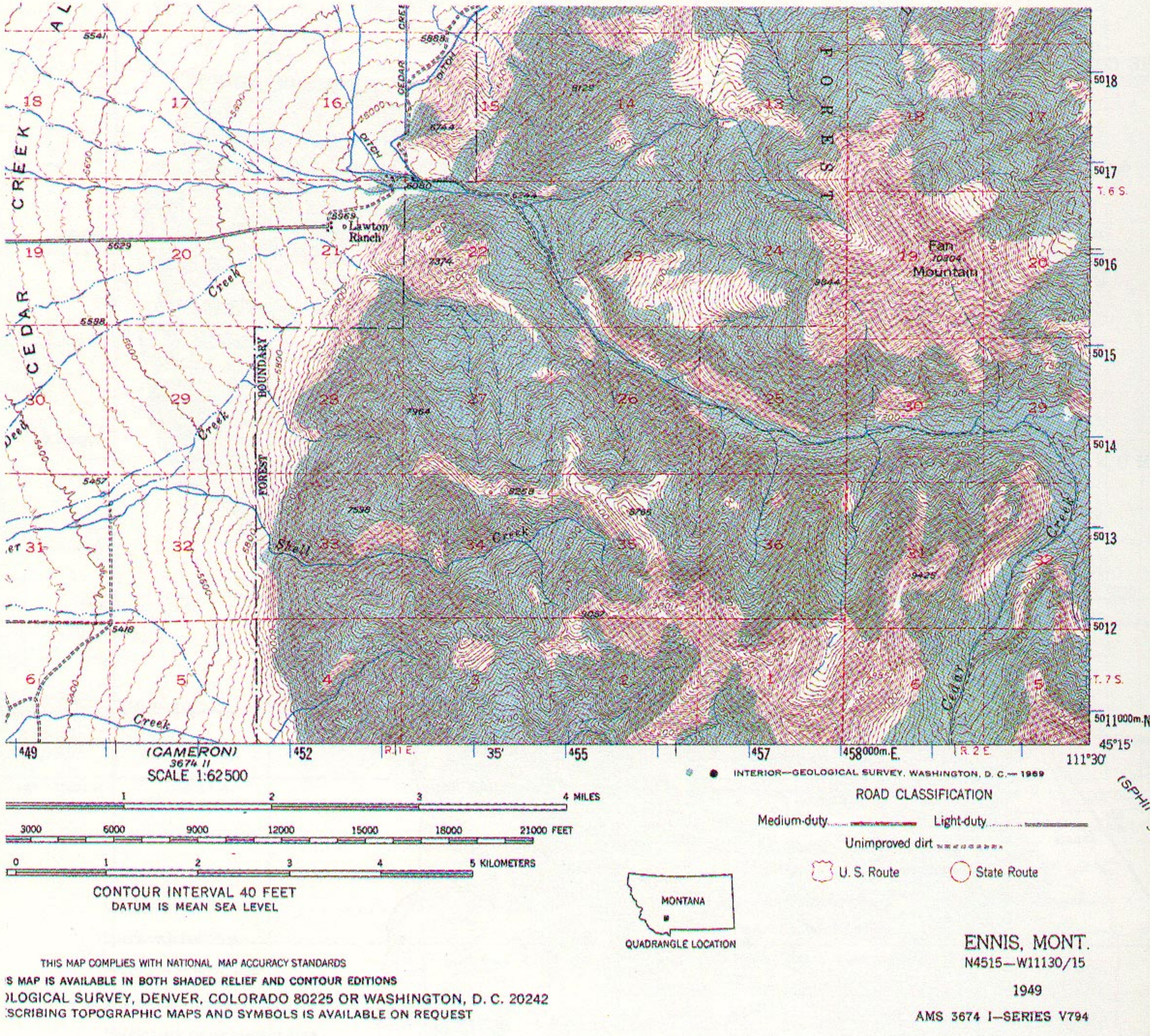
- 31 principal meridians and base lines
- Townships numbered N or S of baseline
- Ranges numbered E or W of principal meridian
- Initially no allowance made for convergence and accumulated error, later error placed in N and W portions of township

- Each 6-mile square township divided into 36 sections.
- Each section may be subdivided into smaller parcels
- Homestead Act of 1862 granted 160 acres to head of household was based on PLSS.
- Periodically, due to convergence, township lines were adjusted. Every 24 miles from the baseline a standard parallel or correction line is used to correct for longitudinal convergence.

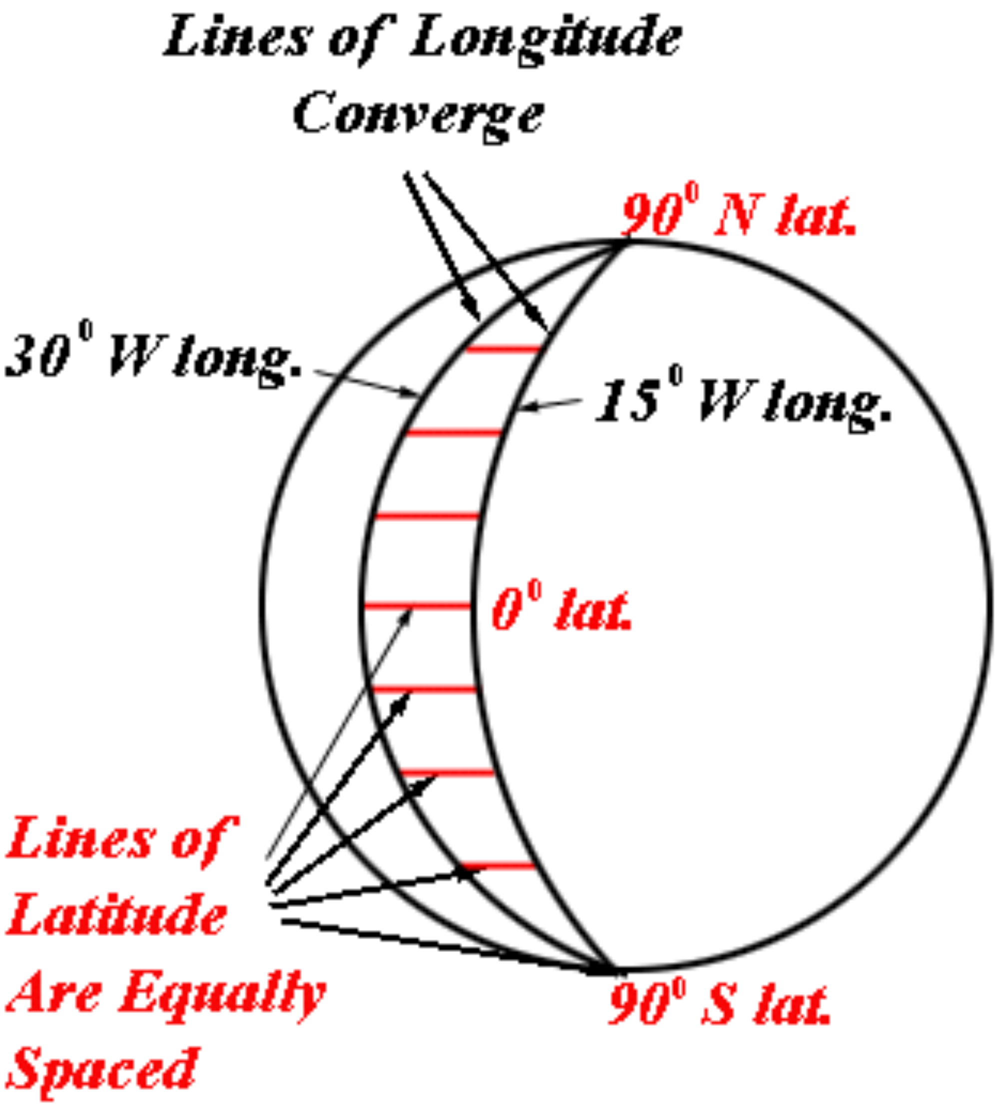


Typical Section Subdivisions

FIGURE 104. Portion of a 15-minute topographic map at scale of 1:62,500. (Ennis, Mont., quadrangle.)



Unfortunately, the world is not square!



How surveys were performed

- Lines oriented to true north
  - Observed the sun using the altitude method where time was not critical
  - Based on known latitude, sun’s declination and sun’s altitude at time of observation.
- Research by Mikhail and Anderson yielded a probable standard deviation of +/- 10-15 seconds for these azimuths.

More considerations

- Terrestrial observations done with respect to the local direction of gravity.
  - in other words, the latitude, longitude and azimuth determined astronomically depend on the local direction of gravity and hence refer to the geoid.
- Astronomical observations were not reduced to a reference ellipsoid.
  - Precision of the measurements made many issues moot.

Squares

- Requirement that east-west lines cross at right angles leads to concept of rhumb line.
  - Rhumb line is a line on the earth's surface that intersects all meridians at the same angle i.e a line of constant azimuth.
  - Parallels of latitude are special rhumb lines that meet each meridian at right angles and remain equidistant from poles.

Quadrangle

- The township defined by law as two converging straight lines forming the east west boundaries is not square.
- North boundary is shorter than south due to convergence.
- “Correction lines” are used to address this problem (placing a rectangular system over a large area).

Convergence

- Convergence is a systematic error because it can be quantified.
- On the ellipsoid we can more precisely estimate convergence.
- It is not the case that forward and backward azimuths differ by 180 degrees exactly.
  - back az = fwd az + 180d + convergence

USPLSS characteristics

- Measured distances reduced to horizontal NOT to reference ellipsoid.
- Astronomic observations used for orientation. System non-orthogonal due to convergence or meridians
- East-west lines of a township are not parallel.
- Because of convergence a perfect traverse would not close.